

# THE VISUAL ANGLE METRIC AND QUASIREGULAR MAPS

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**ABSTRACT.** The distortion of distances between points under maps is studied. We first prove a Schwarz-type lemma for quasiregular maps of the unit disk involving the visual angle metric. Then we investigate conversely the quasiconformality of a bilipschitz map with respect to the visual angle metric on convex domains. For the unit ball or half space, we prove that a bilipschitz map with respect to the visual angle metric is also bilipschitz with respect to the hyperbolic metric. We also obtain various inequalities relating the visual angle metric to other metrics such as the distance ratio metric and the quasihyperbolic metric.

**KEYWORDS.** the visual angle metric, the hyperbolic metric, Lipschitz map, quasiregular map

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## 1. INTRODUCTION

One of the main problems of geometric function theory is to study the way in which maps distort distances between points. The standard method to discuss this problem is to study uniform continuity of maps between suitable metric spaces. For example, one of the cornerstones of geometric function theory, the Schwarz Lemma originally formulated for bounded analytic functions of the unit disk, has been extended to several other classes of functions and to several metrics other than the Euclidean metric.

Let  $G \subsetneq \mathbb{R}^n$  be a domain and  $x, y \in G$ . The *visual angle metric* is defined as

$$v_G(x, y) = \sup_{z \in \partial G} \angle(x, z, y),$$

where  $\partial G$  is not a proper subset of a line. This paper is based on our earlier paper [KLVW], where we introduced this metric and studied some of the basic properties of the visual angle metric. In particular, we gave some estimates for the visual angle metric in terms of the hyperbolic metric in the case when the domain is either the unit ball or the upper half space. Our goal here is to study how the visual angle metric behaves under quasiconformal maps. The main result is the following theorem.

**Theorem 1.1.** *If  $f : \mathbb{B}^2 \rightarrow \mathbb{R}^2$  is a non-constant  $K$ -quasiregular map with  $f\mathbb{B}^2 \subset \mathbb{B}^2$ , then*

$$v_{\mathbb{B}^2}(f(x), f(y)) \leq C(K) \max\{v_{\mathbb{B}^2}(x, y), v_{\mathbb{B}^2}(x, y)^{1/K}\}$$

*for all  $x, y \in \mathbb{B}^2$ , where  $C(K) = 2 \cdot 4^{1-1/K}$  and  $C(1) = 2$ .*

**Remark 1.2.** It is clear that the visual angle metric is similarity invariant. It is not difficult to show that it fails to be Möbius invariant. However, by Lemma 3.5 the visual angle metric is not changed by more than a factor 2 under the Möbius transformations of  $G$  onto  $G'$  for  $G, G' \in \{\mathbb{B}^n, \mathbb{H}^n\}$ . For Möbius transformations of the unit disk onto itself, we know that the best constant in place of  $C(K)$  is 2 by [KLVW, Theorem 1.2]. Therefore, we see that the constant  $C(K)$  is asymptotically sharp when  $K \rightarrow 1$ .

Moreover, we prove

**Theorem 1.3.** *Let  $G_1, G_2$  be proper convex subdomains of  $\mathbb{R}^n$ . Let  $f : G_1 \rightarrow G_2 = f(G_1)$  be an  $L$ -bilipschitz map with respect to the visual angle metric. Then  $f$  is quasiconformal and with linear dilatation at most  $4L^2$ .*

**Theorem 1.4.** *For  $G_1, G_2 \in \{\mathbb{B}^n, \mathbb{H}^n\}$ , let  $f : G_1 \rightarrow G_2$  be a bilipschitz map with respect to the visual angle metric. Then  $f$  is also a bilipschitz map with respect to the hyperbolic metric.*

We also prove various inequalities relating the visual angle metric to other metrics such as the distance ratio metric and the quasihyperbolic metric. Agard and Gehring [AG, Theorems 2 and 3] have studied the transformation of angles under quasiconformal maps. Our results differ from their work, because in our case the vertex of the angle is on the boundary of the domain of definition of the mapping.

## 2. PRELIMINARIES

**2.1. Notation.** Throughout this paper we will discuss domains  $G \subset \mathbb{R}^n$ , i.e., open and connected subsets of  $\mathbb{R}^n$ . For  $x, y \in G$ , the Euclidean distance between  $x$  and  $y$  is denoted by  $|x - y|$  or  $d(x, y)$ , as usual. The notation  $d(x, \partial G)$  or  $d(x)$  for abbreviation, stands for the distance from the point  $x$  to the boundary  $\partial G$  of the domain  $G$ .

The Euclidean  $n$ -dimensional ball with center  $z$  and radius  $r$  is denoted by  $\mathbb{B}^n(z, r)$ , and its boundary sphere by  $S^{n-1}(z, r)$ . In particular,  $\mathbb{B}^n(r) = \mathbb{B}^n(0, r)$ ,  $S^{n-1}(r) = S^{n-1}(0, r)$ , and  $\mathbb{B}^n = \mathbb{B}^n(0, 1)$ ,  $S^{n-1} = S^{n-1}(0, 1)$ .

The upper Lobachevsky-Poincaré  $n$ -dimensional half space (as a set) is denoted by  $\mathbb{H}^n = \{(z_1, z_2, \dots, z_n) \in \mathbb{R}^n : z_n > 0\}$ . For  $t \in \mathbb{R}$  and  $a \in \mathbb{R}^n \setminus \{0\}$ , we denote a hyperplane in  $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$  by  $P(a, t) = \{x \in \mathbb{R}^n : x \cdot a = t\} \cup \{\infty\}$ .

Given a vector  $u \in \mathbb{R}^n \setminus \{0\}$  and a point  $x \in \mathbb{R}^n$ , the line passing through  $x$  with direction vector  $u$  is denoted by  $L(x, u) = \{x + tu : t \in \mathbb{R}\}$  and the ray starting at  $x$  with direction  $u$  is  $\text{ray}(x, u) = \{x + tu : t > 0\}$ . Given two points  $x$  and  $y$ , the segment between them is denoted by  $[x, y] = \{(1 - t)x + ty : 0 \leq t \leq 1\}$ .

Given three distinct points  $x, y$  and  $z \in \mathbb{R}^n$ , the notation  $\angle(x, z, y)$  means the angle in the range  $[0, \pi]$  between the segments  $[x, z]$  and  $[y, z]$ .

Let  $G$  be a set for which a metric  $m_G$  is defined. We define the open  $m$ -ball  $B_m(x, t)$  with center  $x$  and radius  $t$ , and the corresponding boundary  $m$ -sphere  $S_m(x, t)$ , in  $m$ -metric, as the set

$$B_m(x, t) = \{z \in G : m_G(x, z) < t\}$$

and

$$S_m(x, t) = \{z \in G : m_G(x, z) = t\},$$

respectively.

**2.2. Visual angle metric.** The visual angle metric has the following monotonicity property: if  $G_1, G_2$  are proper subdomains of  $\mathbb{R}^n$  and  $x, y \in G_1 \subset G_2$ , then  $v_{G_1}(x, y) \geq v_{G_2}(x, y)$ .

In the unit ball, for the special case  $x \neq 0, y = 0$ , we have the following useful formula

$$(2.3) \quad v_{\mathbb{B}^n}(0, x) = \arcsin |x| \in (0, \pi/2),$$

and for  $|x| = |y| \neq 0$ ,  $\theta = \frac{1}{2}\angle(x, 0, y) \in (0, \pi/2]$ , we have

$$(2.4) \quad v_{\mathbb{B}^n}(x, y) = 2 \arctan \frac{|x| \sin \theta}{1 - |x| \cos \theta}.$$

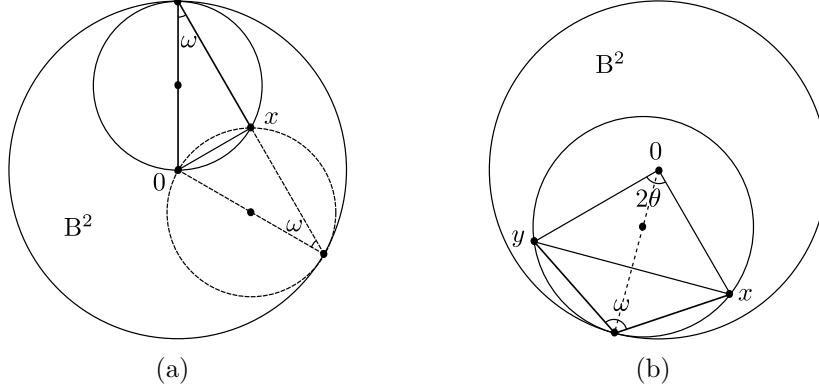


FIGURE 1. Figure in  $\mathbb{B}^2$ : (a) Special case (2.3), where  $y = 0$ . (b) Special case (2.4), where  $|x| = |y|$  and  $\angle(x, 0, y) = 2\theta$ .

It is well-known that the diameter of a Euclidean sphere is twice of its radius, but this is not the case for the visual angle metric in  $\mathbb{B}^n$ .

**Remark 2.5.** Choose  $x \in (0, e_1)$ ,  $M = v_{\mathbb{B}^n}(0, x) \in (0, \pi/2)$ . Then by (2.3),  $B_v(0, M)$  is a Euclidean ball with radius  $|x| = \sin M$ . By (2.4), the diameter of the  $v$ -sphere  $S_v(0, M)$  is

$$v_{\mathbb{B}^n}(-x, x) = 2 \arcsin \frac{|x|}{\sqrt{1 + |x|^2}}.$$

Hence it follows that  $v_{\mathbb{B}^n}(-x, x) < 2M$  and therefore the diameter of  $S_v(0, M)$  is less than twice of the radius  $M$ .

**2.6. Hyperbolic metric.** The explicit formulas of the hyperbolic metric are as follows:

$$(2.7) \quad \text{ch} \rho_{\mathbb{H}^n}(x, y) = 1 + \frac{|x - y|^2}{2d(x, \partial \mathbb{H}^n)d(y, \partial \mathbb{H}^n)}$$

for all  $x, y \in \mathbb{H}^n$  [B, p.35], and

$$(2.8) \quad \text{sh} \frac{\rho_{\mathbb{B}^n}(x, y)}{2} = \frac{|x - y|}{\sqrt{1 - |x|^2} \sqrt{1 - |y|^2}}$$

for all  $x, y \in \mathbb{B}^n$  [B, p.40]. The hyperbolic metric is invariant under Möbius transformations of  $G$  onto  $G'$  for  $G, G' \in \{\mathbb{B}^n, \mathbb{H}^n\}$ . Hyperbolic geodesics are arcs of circles which are orthogonal to the boundary of the domain. The problem of finding the midpoint of a hyperbolic segment has been studied in [VW].

**2.9. Distance ratio metric.** For a proper open subset  $G \subset \mathbb{R}^n$  and for all  $x, y \in G$ , the distance ratio metric  $j_G$  is defined as

$$j_G(x, y) = \log \left( 1 + \frac{|x - y|}{\min\{d(x, \partial G), d(y, \partial G)\}} \right).$$

The distance ratio metric was introduced by Gehring and Palka [GP] and in the above simplified form by Vuorinen [Vu1]. Both definitions are frequently used in the study of hyperbolic type metrics [HIMPS], geometric theory of functions [Vu2], and quasiconformality in Banach spaces [Va1].

**2.10. Quasihyperbolic metric.** Let  $G$  be a proper subdomain of  $\mathbb{R}^n$ . For all  $x, y \in G$ , the quasihyperbolic metric  $k_G$  is defined as

$$k_G(x, y) = \inf_{\gamma} \int_{\gamma} \frac{1}{d(z, \partial G)} |dz|,$$

where the infimum is taken over all rectifiable arcs  $\gamma$  joining  $x$  to  $y$  in  $G$  [GP]. The quasihyperbolic metric has found many applications in geometric function theory [Va1, Vu2]. For some geometric properties of this metric, see [K, L].

It is well known that the following comparison relations hold for  $x, y \in \mathbb{B}^n$ ,

$$(2.11) \quad \frac{1}{2} \rho_{\mathbb{B}^n}(x, y) \leq j_{\mathbb{B}^n}(x, y) \leq \rho_{\mathbb{B}^n}(x, y),$$

$$(2.12) \quad \frac{1}{2} \rho_{\mathbb{B}^n}(x, y) \leq k_{\mathbb{B}^n}(x, y) \leq \rho_{\mathbb{B}^n}(x, y).$$

**2.13. Lipschitz mappings.** Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces. Let  $f : X \rightarrow Y$  be continuous and let  $L \geq 1$ . We say that  $f$  is  $L$ -lipschitz if

$$d_Y(f(x), f(y)) \leq L d_X(x, y), \text{ for } x, y \in X,$$

and  $L$ -bilipschitz if  $f$  is a homeomorphism and

$$d_X(x, y)/L \leq d_Y(f(x), f(y)) \leq L d_X(x, y), \text{ for } x, y \in X.$$

A 1-bilipschitz mapping is called an isometry.

**2.14. Linear dilatation.** Let  $f : G \rightarrow \mathbb{R}^n$  be a continuous discrete function (i.e. the set  $f^{-1}(y), y \in fG$ , consists of isolated points), where  $G$  is a domain of  $\mathbb{R}^n$ . The linear dilatation of  $f$  at a point  $x \in G$  is given by

$$H(x, f) = \lim_{r \rightarrow 0^+} \sup \frac{L(x, f, r)}{l(x, f, r)},$$

where

$$L(x, f, r) = \max_z \{|f(z) - f(x)| : |z - x| = r\},$$

$$l(x, f, r) = \min_z \{|f(z) - f(x)| : |z - x| = r\},$$

for  $r \in (0, d(x))$ .

**2.15. Quasiregular mappings.** Let  $G_1$  and  $G_2$  be domains in  $\mathbb{R}^n$ . A non-constant mapping  $f : G_1 \rightarrow G_2$  is said to be quasiregular if it satisfies the following conditions:

- (1)  $f$  is sense-preserving continuous, discrete, and open;
- (2)  $H(x, f)$  is locally bounded in  $G_1$ ;
- (3) There exists  $a < \infty$  such that  $H(x, f) \leq a$  for a.e.  $x \in G_1 \setminus B_f$ , where  $B_f$  is the branch set of  $f$ .

A quasiregular homeomorphism is called quasiconformal. Hence, a homeomorphism  $f$  is quasiconformal if and only if  $H(x, f)$  is bounded, see [MRV, Theorem 4.13].

**2.16. Automorphisms of  $\mathbb{B}^n$ .** We denote  $a^* = \frac{a}{|a|^2}$  for  $a \in \mathbb{R}^n \setminus \{0\}$ , and  $0^* = \infty$ ,  $\infty^* = 0$ . For a fixed  $a \in \mathbb{B}^n \setminus \{0\}$ , let

$$\sigma_a(z) = a^* + r^2(x - a^*)^*, \quad r^2 = |a|^{-2} - 1$$

be the inversion in the sphere  $S^{n-1}(a^*, r)$  orthogonal to  $S^{n-1}$ . Then  $\sigma_a(a) = 0$ ,  $\sigma_a(a^*) = \infty$ .

Let  $p_a$  denote the reflection in the  $(n-1)$ -dimensional hyperplane  $P(a, 0)$  and define a sense-preserving Möbius transformation by

$$(2.17) \quad T_a = p_a \circ \sigma_a.$$

Then,  $T_a \mathbb{B}^n = \mathbb{B}^n$ ,  $T_a(a) = 0$ , and  $T_a(e_a) = e_a$ ,  $T_a(-e_a) = -e_a$ . For  $a = 0$  we set  $T_0 = id$ , where  $id$  stands for the identity map. It is easy to see that  $(T_a)^{-1} = T_{-a}$ . Let  $\mathcal{GM}(\mathbb{B}^n)$  be the group of Möbius transformations which map  $\mathbb{B}^n$  onto itself. It is well-known that for a given  $g \in \mathcal{GM}(\mathbb{B}^n)$  there is an orthogonal map  $k$  such that  $g = k \circ T_a$ , where  $a = g^{-1}(0)$  [B, Theorem 3.5.1]. For  $x, y \in \mathbb{B}^n$ ,  $|T_x(y)| = \text{th}_{\frac{1}{2}} \rho_{\mathbb{B}^n}(x, y)$  [Vu2, (2.25)].

The next lemma, so-called *monotone form of l'Hôpital's rule*, has found recently numerous applications in proving inequalities. See the extensive bibliography of [AVZ].

**Lemma 2.18.** [AVV, Theorem 1.25] *Let  $-\infty < a < b < \infty$ , and let  $f, g : [a, b] \rightarrow \mathbb{R}$  be continuous on  $[a, b]$ , differentiable on  $(a, b)$ . Let  $g'(x) \neq 0$  on  $(a, b)$ . Then, if  $f'(x)/g'(x)$  is increasing(decreasing) on  $(a, b)$ , so are*

$$\frac{f(x) - f(a)}{g(x) - g(a)} \quad \text{and} \quad \frac{f(x) - f(b)}{g(x) - g(b)}.$$

*If  $f'(x)/g'(x)$  is strictly monotone, then the monotonicity in the conclusion is also strict.*

### 3. THE SCHWARZ-TYPE LEMMA

In this section, we will prove the result related to the Schwarz-type lemma of the visual angle metric under  $K$ -quasiregular maps of the unit disk.

We first study the properties of some special functions involved in the proof of Theorem 1.1. For  $r \in (0, 1)$  and  $K > 0$ , we define the distortion function

$$\varphi_K(r) = \mu^{-1}(\mu(r)/K),$$

where  $\mu(r)$  is the modulus of the planar Grötzsch ring, see [Vu2, Exercise 5.61]. It is clear that  $\varphi_1(r) = r$ . From now on we let  $r' = \sqrt{1 - r^2}$ . Let  $\mathcal{K}$  and  $\mathcal{E}$  be the complete elliptic integral of the first and second kind, respectively. The following derivative formulas of  $\varphi_K(r)$  hold for  $r \in (0, 1)$ ,  $K \in (0, \infty)$  [AVV, (10.6)]:

$$\frac{\partial \varphi_K(r)}{\partial r} = \frac{1}{K} \frac{ss'^2 \mathcal{K}(s)^2}{rr'^2 \mathcal{K}(r)^2},$$

$$\frac{\partial \varphi_K(r)}{\partial K} = \frac{4}{\pi^2 K^2} ss'^2 \mathcal{K}(s)^2 \mu(r),$$

where  $s = \varphi_K(r)$ . By [WZC, Corollary], the function  $\varphi_K(r)$  is concave in  $r$  if  $K > 1$  and convex if  $0 < K < 1$ .

**Lemma 3.1.** *For  $r \in (0, 1)$ ,  $K > 1$ , let  $s = \varphi_K(r)$ . Then the function*

- (1)  $f_1(r) \equiv r^{-1/K} s$  *is strictly decreasing from  $(0, 1)$  onto  $(1, 4^{1-1/K})$ ;*
- (2)  $f_2(r) \equiv \frac{s' \mathcal{K}(s)^2}{r' \mathcal{K}(r)^2}$  *is strictly decreasing from  $(0, 1)$  onto  $(0, 1)$ ;*
- (3)  $f_3(r) \equiv \sqrt{r'} \mathcal{K}(r)$  *is decreasing from  $[0, 1)$  onto  $(0, \pi/2]$ ;*
- (4)  $f_4(r) \equiv \frac{s}{r}$  *is strictly decreasing from  $(0, 1)$  onto  $(1, \infty)$ ;*
- (5)  $f_5(r) \equiv \frac{r}{\arctan(r/r')}$  *is strictly decreasing from  $(0, 1)$  onto  $(2/\pi, 1)$ ;*
- (6)  $f_6(r) \equiv 2\mathcal{E}(r) - r'^2 \mathcal{K}(r)$  *is strictly increasing from  $(0, 1)$  onto  $(\pi/2, 2)$ .*

*Proof.* (1) This is a result from [AVV, Lemma 10.9(1)].

(2) This is a result from [AVV, Lemma 10.7(1)].

(3) This is a result from [AVV, Lemma 3.21(7)].

(4) Since

$$f_4(x) = \frac{s}{r^{1/K}} \cdot \frac{1}{r^{1-1/K}},$$

$f_4$  is strictly decreasing by (1). The limiting values are clear.

(5) Let  $f_{51}(r) = r$  and  $f_{52}(r) = \arctan(r/r')$ . Then  $f_{51}(0^+) = f_{52}(0^+) = 0$ . By differentiation, we have

$$\frac{f'_{51}(r)}{f'_{52}(r)} = r',$$

which is strictly decreasing. Hence, by Lemma 2.18  $f_5$  is strictly decreasing. The limiting value  $f_5(0^+) = 1$  follows by l'Hôpital Rule and  $f_5(1^-) = 2/\pi$  is clear.

(6) This is a result from [AVV, Exercise 3.43(13)].

□

**Lemma 3.2.** *For  $r \in (0, 1)$ ,  $K > 1$ , let  $s = \varphi_K(r)$ . Then the function*

- (1)  $f_1(r) \equiv \frac{\arctan(s/s')}{\arctan(r/r')}$  *is strictly decreasing from  $(0, 1)$  onto  $(1, \infty)$ ;*
- (2)  $f_2(r) \equiv \frac{\arctan(s/s')}{(\arctan(r/r'))^{1/K}}$  *is strictly decreasing from  $(0, 1)$  onto  $((\pi/2)^{1-1/K}, 4^{1-1/K})$ .*

*Proof.* (1) Let  $f_{11}(r) = \arctan(s/s')$  and  $f_{12}(r) = \arctan(r/r')$ . Then  $f_{11}(0^+) = f_{12}(0^+) = 0$ . By differentiation, we have

$$\frac{f'_{11}(r)}{f'_{12}(r)} = \frac{1}{K} \cdot \frac{s}{r} \cdot \frac{s' \mathcal{K}(s)^2}{r' \mathcal{K}(r)^2},$$

which is strictly decreasing by Lemma 3.1(2)(4). Therefore, by Lemma 2.18  $f_1$  is strictly decreasing. The limiting value  $f_1(0^+) = \infty$  follows by l'Hôpital Rule and  $f_1(1^-) = 1$  is clear.

(2) By logarithmic differentiation in  $r$ ,

$$\frac{f'_2(r)}{f_2(r)} = \frac{1}{K} \cdot \frac{1}{r r'^2 \mathcal{K}(r)^2} \left( \frac{s s' \mathcal{K}(s)^2}{\arctan(s/s')} - \frac{r r' \mathcal{K}(r)^2}{\arctan(r/r')} \right).$$

By Lemma 3.1(3)(5), the function  $\frac{r r' \mathcal{K}(r)^2}{\arctan(r/r')}$  is strictly decreasing on  $(0, 1)$ . Since  $s > r$ ,  $f'_2(r) < 0$  and hence  $f_2$  is strictly decreasing. The limiting value  $f_2(1^-) = (\pi/2)^{1-1/K}$  is clear. By Lemma 3.1(1)

$$f_2(0^+) = \lim_{r \rightarrow 0^+} \frac{\arctan(s/s')}{(\arctan(r/r'))^{1/K}} = \lim_{r \rightarrow 0^+} \frac{s/s'}{(r/r')^{1/K}} = \lim_{r \rightarrow 0^+} \frac{s}{r^{1/K}} \cdot \frac{r'^{1/K}}{s'} = 4^{1-1/K}.$$

□

**Lemma 3.3.** For  $K \geq 1$ , let  $r_0 = \frac{\tan 1}{\sqrt{1+\tan^2 1}} \approx 0.841471$  and  $s_0 = \varphi_K(r_0)$ . Then the function

$$f(K) \equiv 4^{1-1/K} \cdot \frac{\arctan(r_0/r'_0)}{\arctan(s_0/s'_0)}$$

is strictly increasing. In particular,

$$(3.4) \quad \max \left\{ 4^{1-1/K}, \frac{\arctan(s_0/s'_0)}{\arctan(r_0/r'_0)} \right\} = 4^{1-1/K}.$$

*Proof.* By logarithmic differentiation in  $K$ ,

$$\frac{f'(K)}{f(K)} = \frac{\log 4}{K^2} (1 - g(K)),$$

where

$$g(K) = \frac{4\mu(r_0)}{\pi^2 \log 4} \cdot \frac{s_0}{\arctan(s_0/s'_0)} \cdot \left( \sqrt{s'_0} \mathcal{K}(s_0) \right)^2.$$

Since  $s_0$  is increasing in  $K$ ,  $g$  is decreasing in  $K$  by Lemma 3.1(3)(5). Then  $g(K) < g(1) = \frac{4\mu(r_0)}{\pi^2 \log 4} \cdot \frac{r_0 r'_0 \mathcal{K}(r_0)^2}{\arctan(r_0/r'_0)} \approx 0.744915 < 1$ . Hence  $f'(K) > 0$ , which implies that  $f$  is strictly increasing. Therefore, (3.4) follows by the monotonicity of  $f$  and  $f(1) = 1$ .  $\square$

**Lemma 3.5.** [KLVW, Theorem 1.1] For  $G \in \{\mathbb{B}^n, \mathbb{H}^n\}$  and  $x, y \in G$ , let  $\rho_G^*(x, y) = \arctan \left( \text{sh} \frac{\rho_G(x, y)}{2} \right)$ . Then

$$\rho_G^*(x, y) \leq v_G(x, y) \leq 2\rho_G^*(x, y).$$

The left-hand side of the inequality is sharp and the constant 2 in the right-hand side of the inequality is the best possible.

**Lemma 3.6.** [Vu2, Theorem 11.2] Let  $f : \mathbb{B}^2 \rightarrow \mathbb{R}^2$  be a non-constant  $K$ -quasiregular mapping with  $f\mathbb{B}^2 \subset \mathbb{B}^2$ . Then for all  $x, y \in \mathbb{B}^2$ ,

$$\text{th} \frac{1}{2} \rho_{\mathbb{B}^2}(f(x), f(y)) \leq \varphi_K \left( \text{th} \frac{1}{2} \rho_{\mathbb{B}^2}(x, y) \right).$$

We are now in a position to prove Theorem 1.1.

*Proof of Theorem 1.1.* By Lemma 3.5 and Lemma 3.6, we get

$$\begin{aligned} A \equiv \frac{v_{\mathbb{B}^2}(f(x), f(y))}{\max \{v_{\mathbb{B}^2}(x, y), v_{\mathbb{B}^2}(x, y)^{1/K}\}} &\leq \frac{2 \arctan(\text{sh} \frac{\rho_{\mathbb{B}^2}(f(x), f(y))}{2})}{\max \left\{ \arctan(\text{sh} \frac{\rho_{\mathbb{B}^2}(x, y)}{2}), (\arctan(\text{sh} \frac{\rho_{\mathbb{B}^2}(x, y)}{2}))^{1/K} \right\}} \\ &\leq \frac{2 \arctan(s/s')}{\max \{ \arctan(r/r'), (\arctan(r/r'))^{1/K} \}}, \end{aligned}$$

where  $r = \text{th} \frac{\rho_{\mathbb{B}^2}(x, y)}{2}$  and  $s = \varphi_K(r)$ . By Lemma 3.2 and Lemma 3.3,

$$A \leq 2 \max \{f_1(r_0), f_2(0^+)\} = 2 \cdot 4^{1-1/K},$$

where  $f_1, f_2$  are as in Lemma 3.2 and  $r_0$  is as in Lemma 3.3.

This completes the proof of Theorem 1.1.  $\square$

In [BV], an explicit form of the Schwarz lemma for quasiregular mappings was given.

**Theorem 3.7.** [BV, Theorem 1.10] *If  $f : \mathbb{B}^2 \rightarrow \mathbb{R}^2$  is a non-constant  $K$ -quasiregular map with  $f\mathbb{B}^2 \subset \mathbb{B}^2$  and  $\rho$  is the hyperbolic metric of  $\mathbb{B}^2$ , then*

$$\rho_{\mathbb{B}^2}(f(x), f(y)) \leq c(K) \max\{\rho_{\mathbb{B}^2}(x, y), \rho_{\mathbb{B}^2}(x, y)^{1/K}\}$$

for all  $x, y \in \mathbb{B}^2$ , where  $c(K) = 2 \operatorname{arth}(\varphi_K(\operatorname{th} \frac{1}{2}))$  and, in particular,  $C(1) = 1$ .

The proof of this theorem involves the following monotonicity of the transcendental function  $\varphi_K(r)$ , see [BV, Lemma 4.8].

**Lemma 3.8.** *For  $K > 1$ , the function*

$$g(r) \equiv \frac{\operatorname{arth}(\varphi_K(r))}{(\operatorname{arth} r)^{1/K}}$$

*is strictly increasing on  $(0, 1)$ .*

There exists, however, a gap in the proof of the above result due to the using of a wrong claim that the function  $r \mapsto \varphi_K(r)/r$  is increasing on  $(0, 1)$ . In fact, by [WZC, Corollary], the function  $\varphi_K(r)$  is concave in  $r$  if  $K > 1$ , and hence the function  $r \mapsto \varphi_K(r)/r$  is decreasing on  $(0, 1)$ , see Lemma 3.1(4). We give a correction for the proof of the monotonicity of the function  $g$  as follows.

*Proof.* Let  $g_1(r) = r\mathcal{K}(r)^2 / \operatorname{arth} r = g_{11}(r)/g_{12}(r)$ , where  $g_{11}(r) = r\mathcal{K}(r)^2$  and  $g_{12}(r) = \operatorname{arth} r$ . Then  $g_{11}(0) = g_{12}(0) = 0$ , and

$$\frac{g'_{11}(r)}{g'_{12}(r)} = \mathcal{K}(r)(2\mathcal{E}(r) - r'^2\mathcal{K}(r)),$$

which is strictly increasing by Lemma 3.1(6) and implies that  $g_1(r)$  is also strictly increasing by Lemma 2.18. Let  $s = \varphi_K(r)$ . Then we have that

$$(3.9) \quad \frac{s\mathcal{K}(s)^2}{\operatorname{arth} s} - \frac{r\mathcal{K}(r)^2}{\operatorname{arth} r} > 0$$

since  $s > r$  for all  $K > 1$  and  $0 < r < 1$ . By logarithmic differentiation, we get that

$$\frac{g'(r)}{g(r)} = \frac{1}{K} \frac{1}{rr'^2\mathcal{K}(r)^2} \left( \frac{s\mathcal{K}(s)^2}{\operatorname{arth} s} - \frac{r\mathcal{K}(r)^2}{\operatorname{arth} r} \right) > 0,$$

which implies that for given  $K > 1$ , the function  $g$  is strictly increasing on  $(0, 1)$ . □

The following result gives the sharp distortion of the distance ratio metric and the quasihyperbolic metric under quasiregular mappings of the unit disk into itself. A similar result has been obtained in [KVZ, Theorem 1.8] for the higher dimensional case.

**Corollary 3.10.** *If  $f : \mathbb{B}^2 \rightarrow \mathbb{R}^2$  is a non-constant  $K$ -quasiregular map with  $f\mathbb{B}^2 \subset \mathbb{B}^2$ , then*

$$j_{\mathbb{B}^2}(f(x), f(y)) \leq 2c(K) \max\{j_{\mathbb{B}^2}(x, y), j_{\mathbb{B}^2}(x, y)^{1/K}\}$$

and

$$k_{\mathbb{B}^2}(f(x), f(y)) \leq 2c(K) \max\{k_{\mathbb{B}^2}(x, y), k_{\mathbb{B}^2}(x, y)^{1/K}\}$$

for all  $x, y \in \mathbb{B}^2$ , where  $c(K)$  is the same as in Theorem 3.7.



*Proof.* For the distance ratio metric, we combine the comparison relation (2.11) with Theorem 3.7 to get that, for  $x, y \in \mathbb{B}^2$  and  $x \neq y$ ,

$$\begin{aligned} \frac{j_{\mathbb{B}^2}(f(x), f(y))}{\max\{j_{\mathbb{B}^2}(x, y), j_{\mathbb{B}^2}(x, y)^{1/K}\}} &\leq \frac{\rho_{\mathbb{B}^2}(f(x), f(y))}{\max\{\frac{1}{2}\rho_{\mathbb{B}^2}(x, y), \frac{1}{2^{1/K}}\rho_{\mathbb{B}^2}(x, y)^{1/K}\}} \\ &= \frac{2\rho_{\mathbb{B}^2}(f(x), f(y))}{\max\{\rho_{\mathbb{B}^2}(x, y), 2^{1-1/K}\rho_{\mathbb{B}^2}(x, y)^{1/K}\}} \\ &\leq \frac{2\rho_{\mathbb{B}^2}(f(x), f(y))}{\max\{\rho_{\mathbb{B}^2}(x, y), \rho_{\mathbb{B}^2}(x, y)^{1/K}\}} \\ &\leq 2c(K). \end{aligned}$$

By use of the inequality (2.12) and a similar argument with the distance ratio metric, we get the result for the quasihyperbolic metric.  $\square$

#### 4. CONVEX DOMAINS AND BILIPSCHITZ MAPS

The following theorem shows the comparison of the visual angle metric and the distance ratio metric on convex domains, which is the main tool to prove Theorem 1.3.

**Theorem 4.1.** *Let  $G$  be a proper convex subdomain of  $\mathbb{R}^n$ . Let  $x, y \in G$  and  $t = e^{j_G(x, y)} - 1$ . Then*

$$\arcsin \frac{t}{t+2} \leq v_G(x, y) \leq 2 \arcsin \frac{t}{\sqrt{4+t^2}}.$$

*The left-hand side of the inequality is sharp and the constant 2 in the right-hand side of the inequality is the best possible.*

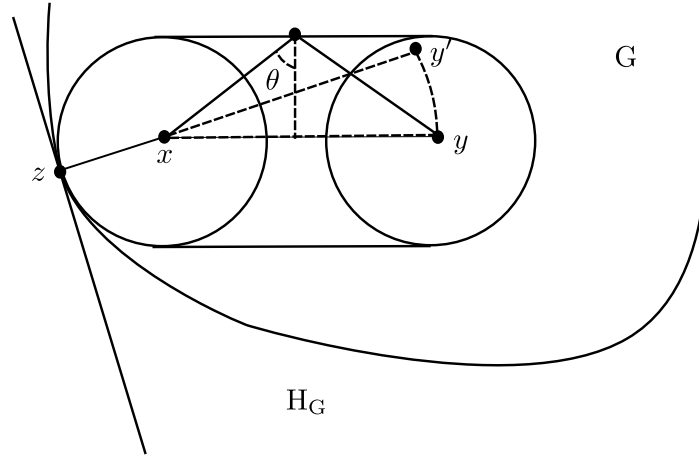


FIGURE 2

*Proof.* Without loss of generality, we may assume that  $d(x) \leq d(y)$ . Then  $t = \frac{|x-y|}{d(x)}$ . Choose  $z \in \partial G$  such that  $d(x) = |x-z|$ . Let  $y' \in \text{ray}(x, x-z)$  with  $|y-x| = |y'-x|$ . Let  $H_G$ ,  $x \in H_G$ , be the half space whose boundary is orthogonal to  $[x, z]$  at the point  $z$ . It is easy to see that  $G \subset H_G$ . Let  $B_{xy}$  be the convex hull of  $B^n(x, d(x)) \cup B^n(y, d(x))$ , see Figure 2. By the monotonicity property of  $v_G$ , we have

$$v_{H_G}(x, y) \leq v_G(x, y) \leq v_{B_{xy}}(x, y),$$

where

$$v_{H_G}(x, y) \geq v_{H_G}(x, y') = \arcsin \frac{|x - y|/2}{d(x) + |x - y|/2} = \arcsin \frac{t}{t + 2},$$

and

$$v_{B_{xy}}(x, y) = 2\theta = 2 \arcsin \frac{|x - y|/2}{\sqrt{d(x)^2 + (|x - y|/2)^2}} = 2 \arcsin \frac{t}{\sqrt{4 + t^2}}.$$

For the sharpness of the left-hand side of the inequality, we consider the domain  $\mathbb{H}^n$  and two points  $x, y \in \mathbb{H}^n$  with  $L(x, y - x)$  perpendicular to the boundary  $\partial\mathbb{H}^n$ .

For the sharpness of the right-hand side of the inequality, we consider the domain  $\mathbb{H}^n$  and two points  $x, y \in \mathbb{H}^n$  with  $L(x, y - x)$  parallel to the boundary  $\partial\mathbb{H}^n$ .

This completes the proof.  $\square$

*Proof of Theorem 1.3.* Let  $t = e^{j_{G_1}(x, y)} - 1$ . Given  $\varepsilon > 0$ , for small enough  $t > 0$ , by Theorem 4.1 we have

$$\frac{t}{2(1 + \varepsilon)} \leq \arcsin \frac{t}{t + 2} \leq v_{G_1}(x, y) \leq 2 \arcsin \frac{t}{\sqrt{4 + t^2}} \leq 2 \arcsin \frac{t}{2} \leq (1 + \varepsilon)t.$$

Then we have

$$\begin{aligned} \frac{|f(x) - f(y)|}{|f(x) - f(z)|} &\leq \frac{2(1 + \varepsilon) \min\{d(f(x)), d(f(y))\} v_{G_2}(f(x), f(y))}{\frac{1}{1 + \varepsilon} \min\{d(f(x)), d(f(z))\} v_{G_2}(f(x), f(z))} \\ &\leq \frac{2(1 + \varepsilon)^2 \min\{d(f(x)), d(f(y))\} L v_{G_1}(x, y)}{\min\{d(f(x)), d(f(z))\} v_{G_1}(x, z)/L} \\ &\leq \frac{2(1 + \varepsilon)^2 \min\{d(f(x)), d(f(y))\} L(1 + \varepsilon) \frac{|x - y|}{\min\{d(x), d(y)\}}}{\min\{d(f(x)), d(f(z))\} \frac{1}{2(1 + \varepsilon)L} \frac{|x - z|}{\min\{d(x), d(z)\}}} \\ &= 4(1 + \varepsilon)^4 L^2 \cdot \frac{\min\{d(f(x)), d(f(y))\}}{\min\{d(f(x)), d(f(z))\}} \cdot \frac{\min\{d(x), d(z)\}}{\min\{d(x), d(y)\}} \cdot \frac{|x - y|}{|x - z|} \\ &\rightarrow 4(1 + \varepsilon)^4 L^2, \quad \text{as } |x - y| = |x - z| = r \rightarrow 0. \end{aligned}$$

Since  $\varepsilon$  is arbitrary, we have  $H(x, f) \leq 4L^2$ . Hence  $f$  is quasiconformal and with linear dilatation at most  $4L^2$ .  $\square$

**Remark 4.2.** By Theorem 1.3, a bilipschitz map from the unit ball onto itself with respect to the visual angle metric is a quasiconformal map. However, the converse is not true.

For example, let  $a \in (0, 1)$  and  $z \in \mathbb{B}^n$ . The radial map  $f(z) = z/|z|^{1-a}$  is quasiconformal [Va2, 16.2]. But, if putting  $|x| = |y|$ ,  $2\theta = \angle(x, 0, y) > 0$ , then by (2.4) we have

$$\lim_{x \rightarrow 0} \frac{v_{\mathbb{B}^n}(f(x), f(y))}{v_{\mathbb{B}^n}(x, y)} = \lim_{x \rightarrow 0} \frac{\arctan \frac{|x|^a \sin \theta}{1 - |x|^a \cos \theta}}{\arctan \frac{|x| \sin \theta}{1 - |x| \cos \theta}} = \lim_{x \rightarrow 0} |x|^{-(1-a)} \frac{1 - |x| \cos \theta}{1 - |x|^a \cos \theta} = \infty.$$

**Lemma 4.3.** Let  $L \in [1, \infty)$  and  $\varepsilon \in (0, 1)$ . Then

- (1)  $f_1(r) \equiv \frac{\arcsin r}{\log(1+r)}$  is strictly increasing from  $(0, 1)$  onto  $(1, \pi/\log 4)$ ;
- (2)  $f_2(r) \equiv \frac{\sin(4Lr)}{\sin r}$  is strictly decreasing from  $(0, \frac{\pi}{8L})$  onto  $((\sin \frac{\pi}{8L})^{-1}, 4L)$ ;
- (3)  $f_3(r) \equiv \frac{\operatorname{arth}(4Lr)}{\operatorname{arth} r}$  is strictly increasing from  $(0, \frac{\varepsilon}{4L})$  onto  $(4L, \operatorname{arth} \varepsilon / \operatorname{arth} \frac{\varepsilon}{4L})$ .

*Proof.* (1) Let  $f_1(r) = \frac{f_{11}(r)}{f_{12}(r)}$  with  $f_{11}(r) = \arcsin r$  and  $f_{12}(r) = \log(1+r)$ . It is clear that  $f_{11}(0^+) = f_{12}(0^+) = 0$ . By differentiation,

$$\frac{f'_{11}(r)}{f'_{12}(r)} = \sqrt{\frac{1+r}{1-r}},$$

which is increasing on  $(0, 1)$ . Therefore,  $f_1$  is strictly increasing by Lemma 2.18. The limiting value  $f_1(1^-) = \pi/\log 4$  is clear and  $f_1(0^+) = 1$  by l'Hôpital's Rule.

(2) Let  $f_{21}(r) = \sin(4Lr)$  and  $f_{22}(r) = \sin r$ . It is clear that  $f_{21}(0^+) = f_{22}(0^+) = 0$ . Then

$$\frac{f'_{21}(r)}{f'_{22}(r)} = 4L \frac{\cos(4Lr)}{\cos r} \equiv 4Lh(r).$$

Since

$$\cos^2 r h'(r) = -4L \sin((4L-1)r) - (4L-1) \sin r \cos(4Lr) < 0,$$

by Lemma 2.18,  $f_2$  is strictly decreasing on  $(0, \frac{\pi}{8L})$ . The limiting value  $f_2(\frac{\pi}{8L}) = (\sin \frac{\pi}{8L})^{-1}$  is clear and  $f_2(0^+) = 4L$  by l'Hôpital Rule.

(3) Let  $f_{31}(r) = \operatorname{arth}(4Lr)$  and  $f_{32}(r) = \operatorname{arth} r$ . It is clear that  $f_{31}(0^+) = f_{32}(0^+) = 0$ . Then

$$\frac{f'_{31}(r)}{f'_{32}(r)} = 4L \frac{1-r^2}{1-(4Lr)^2} = \frac{1}{4L} \left( 1 + \frac{(4L)^2 - 1}{1 - (4Lr)^2} \right),$$

which is strictly increasing on  $(0, \frac{\varepsilon}{4L})$ . Therefore,  $f_3$  is strictly increasing on  $(0, \frac{\varepsilon}{4L})$  by Lemma 2.18. The limiting value  $f_3(\frac{\varepsilon}{4L}) = \operatorname{arth} \varepsilon / \operatorname{arth} \frac{\varepsilon}{4L}$  is clear and  $f_3(0^+) = 4L$  by l'Hôpital Rule. □

**Theorem 4.4.** *Let  $G$  be a proper convex subdomain of  $\mathbb{R}^n$ . For all  $x, y \in G$ , there holds*

$$v_G(x, y) \leq ck_G(x, y),$$

where  $c = \frac{\pi}{\log 4} \approx 2.26618$ .

*Proof.* For arbitrary  $x, y \in G$  such that  $x \neq y$ , the quasihyperbolic geodesic segment with end points  $x, y$  is denoted by  $J_k[x, y]$ . Select points  $\{z_i\}_{i=0}^n \in J_k[x, y]$  with  $z_0 = x$ ,  $z_n = y$  such that  $t_i = \frac{|z_{i+1} - z_i|}{\min\{d(z_i), d(z_{i+1})\}} \in (0, 1)$ , where  $i = 0, \dots, n-1$ .

By Theorem 4.1 and Lemma 4.3(1), we have

$$\begin{aligned} v_G(z_i, z_{i+1}) &\leq 2 \arcsin \frac{t_i}{\sqrt{4 + t_i^2}} \leq 2 \arcsin \frac{t_i}{2} \leq \arcsin t_i \\ &\leq \frac{\pi}{\log 4} j_G(z_i, z_{i+1}) \leq \frac{\pi}{\log 4} k_G(z_i, z_{i+1}). \end{aligned}$$

Then

$$v_G(x, y) \leq \sum_{i=0}^{n-1} v_G(z_i, z_{i+1}) \leq \frac{\pi}{\log 4} \sum_{i=0}^{n-1} k_G(z_i, z_{i+1}) = \frac{\pi}{\log 4} k_G(x, y).$$

This completes the proof. □

**Lemma 4.5.** *Let  $f : \mathbb{B}^n \rightarrow \mathbb{B}^n$  be an  $L$ -bilipschitz map with respect to the visual angle metric. Let  $\varepsilon \in (0, 1)$  and  $c(\varepsilon) = \operatorname{arth} \varepsilon / \operatorname{arth} \frac{\varepsilon}{4L}$ . Furthermore, let  $x, y \in \mathbb{B}^n$  and satisfy*

$$(4.6) \quad \operatorname{th} \frac{\rho_{\mathbb{B}^n}(x, y)}{2} \leq \min \left\{ \frac{\varepsilon}{4L}, \sin \frac{\pi}{8L} \right\}.$$

Then

$$\rho_{\mathbb{B}^n}(f(x), f(y)) \leq c(\varepsilon)\rho_{\mathbb{B}^n}(x, y).$$

*Proof.* Let  $x, y \in \mathbb{B}^n$  satisfy (4.6) such that  $x \neq y$ . By Lemma 3.5, it is not difficult to obtain that  $g = T_{f(x)} \circ f \circ T_{-x} : \mathbb{B}^n \rightarrow \mathbb{B}^n$  is a  $4L$ -bilipschitz map with respect to the visual angle metric and  $g(0) = 0$ . Then by (2.3)

$$\arcsin |T_{f(x)}(f(y))| = v_{\mathbb{B}^n}(g(T_x(x)), g(T_x(y))) \leq 4Lv_{\mathbb{B}^n}(T_x(x), T_x(y)) = 4L \arcsin |T_x(y)|.$$

Since  $\arcsin |T_x(y)| = \arcsin \operatorname{th} \frac{\rho(x, y)}{2} \leq \frac{\pi}{8L}$ , by Lemma 4.3(2) we have

$$|T_{f(x)}(f(y))| \leq \sin(4L \arcsin |T_x(y)|) \leq 4L|T_x(y)|.$$

Hence by Lemma 4.3(3)

$$\rho_{\mathbb{B}^n}(f(x), f(y)) \leq 2\operatorname{arth} \left( 4L \operatorname{th} \frac{\rho_{\mathbb{B}^n}(x, y)}{2} \right) \leq c(\varepsilon)\rho_{\mathbb{B}^n}(x, y).$$

□

In the following we will show that for  $G, G' \in \{\mathbb{B}^n, \mathbb{H}^n\}$ , if  $f : G \rightarrow G'$  is a bilipschitz map with respect to the visual angle metric, then  $f$  is bilipschitz with respect to the hyperbolic metric, too.

**Theorem 4.7.** *Let  $f : \mathbb{B}^n \rightarrow \mathbb{B}^n = f(\mathbb{B}^n)$  be an  $L$ -bilipschitz map with respect to the visual angle metric. Then  $f$  is a  $4L$ -bilipschitz map with respect to the hyperbolic metric.*

*Proof.* For arbitrary  $x, y \in \mathbb{B}^n$  such that  $x \neq y$ , the hyperbolic geodesic segment joining the points  $x, y$  is denoted by  $J_\rho[x, y]$ . Select points  $\{z_k\}_{k=0}^n \in J_\rho[x, y]$  with  $z_0 = x$ ,  $z_n = y$  such that  $\rho_{\mathbb{B}^n}(z_k, z_0) > \rho_{\mathbb{B}^n}(z_{k-1}, z_0)$  and

$$\operatorname{th} \frac{\rho_{\mathbb{B}^n}(z_{k-1}, z_k)}{2} \leq \min \left\{ \frac{\varepsilon}{4L}, \sin \frac{\pi}{8L} \right\},$$

where  $k = 1, \dots, n$  and  $\varepsilon \in (0, 1)$  is a constant. By Lemma 4.5,

$$\rho_{\mathbb{B}^n}(f(x), f(y)) \leq \sum_{k=1}^n \rho_{\mathbb{B}^n}(f(z_{k-1}), f(z_k)) \leq c(\varepsilon) \sum_{k=1}^n \rho_{\mathbb{B}^n}(z_{k-1}, z_k) = c(\varepsilon)\rho_{\mathbb{B}^n}(x, y),$$

where  $c(\varepsilon)$  is as in Lemma 4.5. Then, letting  $\varepsilon$  tend to 0, we obtain

$$\rho_{\mathbb{B}^n}(f(x), f(y)) \leq 4L\rho_{\mathbb{B}^n}(x, y).$$

Applying the above argument to  $f^{-1}$ , we get

$$\rho_{\mathbb{B}^n}(f^{-1}(x), f^{-1}(y)) \leq 4L\rho_{\mathbb{B}^n}(x, y)$$

and hence

$$\rho_{\mathbb{B}^n}(f(x), f(y)) \geq \frac{1}{4L}\rho_{\mathbb{B}^n}(x, y).$$

This completes the proof. □

**Corollary 4.8.** *Let  $f : \mathbb{H}^n \rightarrow \mathbb{H}^n = f(\mathbb{H}^n)$  be an  $L$ -bilipschitz map with respect to the visual angle metric. Then  $f$  is a  $16L$ -bilipschitz map with respect to the hyperbolic metric.*

*Proof.* Let  $g = g_1 \circ f \circ g_2$  where  $g_1 : \mathbb{H}^n \rightarrow \mathbb{B}^n$  and  $g_2 : \mathbb{B}^n \rightarrow \mathbb{H}^n$  are two Möbius transformations. Then  $g : \mathbb{B}^n \rightarrow \mathbb{B}^n$  is a  $4L$ -bilipschitz map with respect to the visual angle metric by Lemma 3.5.

Since  $f = g_1^{-1} \circ g \circ g_2^{-1}$ , by Theorem 4.7 we conclude that  $g$  is a  $16L$ -bilipschitz map with respect to the hyperbolic metric. □

Similarly, we have the following two corollaries.

**Corollary 4.9.** *Let  $f : \mathbb{H}^n \rightarrow \mathbb{B}^n = f(\mathbb{H}^n)$  be an  $L$ -bilipschitz map with respect to the visual angle metric. Then  $f$  is an  $8L$ -bilipschitz map with respect to the hyperbolic metric.*

**Corollary 4.10.** *Let  $f : \mathbb{B}^n \rightarrow \mathbb{H}^n = f(\mathbb{B}^n)$  be an  $L$ -bilipschitz map with respect to the visual angle metric. Then  $f$  is an  $8L$ -bilipschitz map with respect to the hyperbolic metric.*

*Proof of Theorem 1.4.* The result immediately follows from Theorem 4.7, Corollary 4.8, Corollary 4.9 and Corollary 4.10.  $\square$

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## REFERENCES

- [AG] S. B. AGARD AND F. W. GEHRING, Angles and quasiconformal mappings. *Proc. London Math. Soc.* **14a** (1965), 1–21.
- [AVV] G. D. ANDERSON, M. K. VAMANAMURTHY, AND M. VUORINEN, *Conformal Invariants, Inequalities and Quasiconformal Maps*, Canadian Mathematical Society Series of Monographs and Advanced Texts, a Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, 1997.
- [AVZ] G. D. ANDERSON, M. VUORINEN, AND X.-H. ZHANG, *Topics in special functions III*, Analytic Number Theory, Approximation Theory and Special Functions, ed. by K. Alladi, G. Milovanovic, and M. Th. Rassias, 39–85, Springer-Verlag, 2014, arXiv math 1209.1696 [math.CA].
- [B] A. F. BEARDON, *The Geometry of Discrete Groups*, Graduate Texts in Mathematics **91**, Springer-Verlag, New York, 1983.
- [BV] B. A. BHAYO AND M. VUORINEN, *On Mori’s theorem for quasiconformal maps in the  $n$ -space*, *Trans. Amer. Math. Soc.* **363** (2011), 5703–5719.
- [GP] F. W. GEHRING AND B. P. PALKA, *Quasiconformally homogeneous domains*, *J. Analyse Math.* **30** (1976), 172–199.
- [HIMPS] P. HÄSTÖ, Z. IBRAGIMOV, D. MINDA, S. PONNUSAMY, AND S. K. SAHOO, *Isometries of some hyperbolic-type path metrics, and the hyperbolic medial axis*, In the tradition of Ahlfors-Bers. IV, 63–74, *Contemp. Math.* **432**, Amer. Math. Soc., Providence, RI, 2007.
- [K] R. KLÉN, *On hyperbolic type metrics*, Dissertation, University of Turku, Ann. Acad. Sci. Fenn. Math. Diss. No. **152** (2009), 49 pp.
- [KLVW] R. KLÉN, H. LINDÉN, M. VUORINEN, AND G.-D. WANG, *The visual angle metric and Möbius transformations*, *Comput. Methods Funct. Theory*, **14** (2014), 577–608, DOI 10.1007/s40315-014-0075-x,
- [KVZ] R. KLÉN, M. VUORINEN, AND X.-H. ZHANG, *Quasihyperbolic metric and Möbius transformations*, *Proc. Amer. Math. Soc.* **142** (2014), 311–322.
- [L] H. LINDÉN, *Quasihyperbolic geodesics and uniformity in elementary domains*, Dissertation, University of Helsinki, Ann. Acad. Sci. Fenn. Math. Diss. No. **146** (2005), 50 pp.
- [MRV] O. MARTIO, S. RICKMAN, AND J. VÄISÄLÄ, *Definitions for quasiregular mappings*, *Ann. Acad. Sci. Fenn. Ser. A I* **448** (1969), 1–40.
- [Va1] J. VÄISÄLÄ, *The free quasiworld. Freely quasiconformal and related maps in Banach spaces*, *Quasiconformal geometry and dynamics* (Lublin, 1996), 55–118, Banach Center Publ. **48**, Polish Acad. Sci., Warsaw, 1999.
- [Va2] J. VÄISÄLÄ, *Lectures on  $n$ -Dimensional Quasiconformal Mappings*, *Lecture Notes in Mathematics* **229**, Springer-Verlag, Berlin, 1971.
- [Vu1] M. VUORINEN, *Conformal invariants and quasiregular mappings*, *J. Analyse Math.* **45** (1985), 69–115.

- [Vu2] M. VUORINEN, *Conformal Geometry and Quasiregular Mappings*, Lecture Notes in Mathematics **1319**, Springer-Verlag, Berlin, 1988.
- [VW] M. VUORINEN AND G.-D. WANG, *Bisection of geodesic segments in hyperbolic geometry*, Complex Analysis and Dynamical Systems V, Contemp. Math. **591**, Amer. Math. Soc., Providence, RI, 2013, pp. 273-290.
- [WZC] G.-D. WANG, X.-H. ZHANG, AND Y.- M. CHU, *A Hölder mean inequality for the Hersch-Pfluger distortion function (in Chinese)*, Sci. Sin. Math. **40** (2010), 783–786.

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